

calculate explicit form of quintuple self convolution of the rectangular window function.

The quintuple self convolution of the rectangular window function (convolution of the rectangular function with itself five times) results in a piecewise polynomial function of degree 4, often referred to as a 4th-degree B-spline.

For the rectangular window defined as:

$$\text{rect}(t) = \begin{cases} 1 & |t| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

the n-times self convolution (with n=5) has support on the interval $[-n/2, n/2]$, here $[-2.5, 2.5]$, and is expressed by a piecewise polynomial of degree $n-1=4$.

Explicitly, the quintuple convolution $f(t)$ can be written as a scaled version of the 4th-degree B-spline, which is the normalized n-fold convolution of the rectangular function:

$$f(t) = (\text{rect} * \text{rect} * \text{rect} * \text{rect} * \text{rect})(t)$$

with

$$f(t) = \frac{1}{24} \times \begin{cases} 0 & |t| > 2.5 \\ (t+2.5)^4 & -2.5 \leq t < -1.5 \\ -4(t+1.5)^4 + 5(t+2.5)^4 & -1.5 \leq t < -0.5 \\ 6(t+0.5)^4 - 20(t+1.5)^4 + 15(t+2.5)^4 & -0.5 \leq t < 0.5 \\ -4(t-0.5)^4 + 15(t+0.5)^4 - 20(t+1.5)^4 + 5(t+2.5)^4 & 0.5 \leq t < 1.5 \\ (t-1.5)^4 - 4(t-0.5)^4 + 6(t+0.5)^4 - 4(t+1.5)^4 + (t+2.5)^4 & 1.5 \leq t \leq 2.5 \\ 0 & \text{otherwise} \end{cases}$$

This piecewise polynomial gives the quintuple convolution shape as a quartic spline with smooth transitions and support on $[-2.5, 2.5]$.

This formula matches the well-known B-spline of degree 4 generated by five self convolutions of the rectangular window.

Let me know if you want a detailed derivation or visual representation.

this seems to be non-symmetric result. Is it not in error ?

The quintuple convolution of a rectangular window function is indeed symmetric, and the nonsymmetric form given earlier was in error.

Why the convolution of a rectangular window is symmetric

The rectangular window $\text{rect}(t)$ is an even function: $\text{rect}(-t) = \text{rect}(t)$.

The convolution of two even functions is also even (symmetric around zero).

Since the rectangular window is convolved with itself an odd number of times (five), the resulting function must be symmetric in t .

Correcting the explicit form

The quintuple convolution corresponds to the 4th-degree B-spline, which is symmetric about zero, supported on $[-2.5, 2.5]$. Its explicit form is a piecewise degree-4 polynomial symmetric around $t=0$.

A symmetric representation shifts t to the center of the support so the polynomial pieces depend on $|t|$ rather than asymmetric shifts. For example, the known explicit form for the 5-fold convolution of the rect function (normalized B-spline) is:

$$f(t) = \frac{1}{24} \begin{cases} (2.5 - |t|)^4 & |t| \leq 2.5 \\ 0 & \text{otherwise} \end{cases}$$

with polynomial pieces constructed to ensure smooth continuity and symmetry. It is standard to express:

$$f(t) = \frac{1}{24} \sum_{k=1}^5 (-1)^k \binom{5}{k} \left(t + \frac{5}{2} - k\right)^4$$